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Remind: $f : \mathcal{X} \to B$ proper flat

G-equivariant

$\forall K$ compact

$B(\mathbb{R}) \subset K \subset \Omega \subset B(\mathbb{C})$

$\forall m \geq 0 \quad \forall b_1 \ldots b_m \in K$

$\forall r \geq 0 \quad \forall \Omega$ G-stable Stein

$\forall u : \Omega \to \mathcal{X}$ sect. holo

G-equivariant de f sur Ω

$\exists s_n : \Omega \to \mathcal{X}$ section alg.

t.q. $j^r s_n(b_i) = j^r u(b_i)$

et $s_n|_K \xrightarrow{\text{uniforme}} u|_K$

On dit que f est TIGHT

Invariance birationnelle de la propriété tight

Théorème :

X, X' régulière

$$\begin{tikzcd} X' \arrow[rr, dashed, "g"] \arrow[dr, "f'"'] & & X \arrow[dl, "f"] \\ & B & \end{tikzcd}$$

appl. birat t.q. $f \circ g = f'$

Si f tight $\Longrightarrow$ f' tight

Preuve :

HIRONAKA's résol. des singularité

$$\begin{tikzcd} & \tilde{X} \arrow[dl] \arrow[dr] & \\ X' \arrow[rr, dashed, "g"] \arrow[dr, "f'"'] & & X \arrow[dl, "f"] \\ & B & \end{tikzcd}$$

1

$$\begin{tikzcd}
\tilde{X} \arrow[rr, "g"] \arrow[dr, "\tilde{f}"'] & & X \arrow[dl, "f"] \\
& B &
\end{tikzcd}$$

g birat. morphism proper

$f \circ g = \tilde{f}$

$\tilde{f}$ tight $\Rightarrow$ f tight

2

$$\begin{tikzcd}
E \arrow[rr] \arrow[d, phantom, sloped, "\subset"] & & Z \arrow[d, phantom, sloped, "\subset"] \\
\tilde{X} \arrow[rr, "g"] \arrow[dr, "\tilde{f}"'] & & X \arrow[dl, "f"] \\
& B &
\end{tikzcd}$$

Z smooth subvariety

g birat morphism

$f \circ g = \tilde{f}$

f tight $\Rightarrow$ $\tilde{f}$

1

$$\begin{tikzcd} E \subset \tilde{X} \arrow[rr, "g"] \arrow[dr, "\tilde{f}"'] & & X \supset Z \arrow[dl, "f"] \\ & B & \end{tikzcd}$$

g birat. morphism

$f \circ g = \tilde{f}$

$\tilde{f}$ tight $\Longrightarrow$ f tight

$u : \Omega \longrightarrow X$

K compact

Ω STEIN

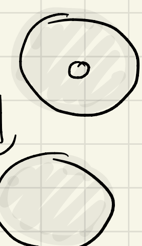
important

- no connected components of $u(\Omega) \subset Z$ (assume this)

- Define the strict transform of u

DEF. because $u(\Omega) \not\subset Z$ hyp. and $g^{-1} \circ u_x$

$$\begin{tikzcd} \tilde{X} \arrow[d, "g"'] & & \mathrm{Spec}(\mathrm{Frac}(\mathcal{O}_{\Omega,x})) \arrow[ll] \arrow[d] \\ X & & \mathrm{Spec}(\mathcal{O}_{\Omega,x}) \arrow[ll, "u_x"] \arrow[ull, dashed, "\exists!\, \tilde{u}_x"'] \end{tikzcd}$$



$\forall K$ compact $g^{-1}(K)$ is compact

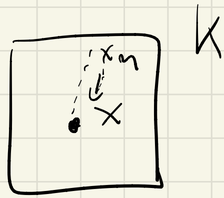
$\boxed{\dim \Omega = 1}$

$x \in \Omega$

Valuative criterion of properness

$$\begin{array}{ccc} \tilde{V} \subseteq \tilde{X} \\ h \downarrow \quad \downarrow g \\ V \subseteq X \end{array}$$

$g^{-1}(K)$ compact $\forall$ compact K



$g^{-1}(x_n)$ in $g^{-1}(K)$

$\exists$ sottosuccessione convergente a $g^{-1}(x)$, I take this as image of $\tilde{\mathcal{U}}_x$

! because of Hausdorff (!)

$$\tilde{u} : \Omega \longrightarrow \tilde{X}$$

$$K \supset \{b_1 \ldots b_m\} \qquad r \geq 0$$

$$\tilde{f} \text{ tight} \Longrightarrow \exists\, \tilde{S}_m$$

$$j^r \tilde{S}_m(b_i) = j^r \tilde{u}(b_i)$$

$$\tilde{S}_m|_K \longrightarrow \tilde{u}|_K$$

$$u = g \circ \tilde{u} \qquad S_m = g \circ \tilde{S}_m$$

$$\begin{aligned}
j^r(g \circ \tilde{u}) &= j^r g \circ j^r \tilde{u} \\
&= j^r g \circ j^r \tilde{S}_m \\
&= j^r(g \circ \tilde{S}_m)
\end{aligned}$$

$$j^r(g\circ\tilde{u}) = j^r u = j^r S_m$$

stesse derivate r-esime

(=) calcola la derivata della composta

$$d\left(g\circ\tilde{u}(t)\right) = g'(\tilde{u}(t))\cdot\tilde{u}'(t)$$

$$d\left(g\circ\tilde{s}_n(t)\right) = g'(s_n(t))\cdot s_n'(t)$$

$$\| $$

$$d(s_n)$$

etc.....

regle de la chaine

Rimane da spiegare cosa succede in $u(\Omega)\cap Z$ e xchè possiamo supporre no c.c. of $u(\Omega)$ che in $E(\mathbb{C})$

alla fine

2

$$\begin{tikzcd} E \subset \tilde{X} \arrow[r, "g"] \arrow[dr, "\tilde{f}"'] & X \supset Z \arrow[d, "f"] \\ \Omega \arrow[u, "\tilde{u}"] & B \end{tikzcd}$$

g birat morphism

$f \circ g = \tilde{f}$

f tight $\Longrightarrow$ $\tilde{f}$ tight

$K, b_1 \dots b_m, \; r \geq 0, \; \tilde{u} : \Omega \to \tilde{X}$

- no c.c. of $\tilde{u}(\Omega)$ are in E
- $\tilde{u}(\Omega) \cap E(\mathbb{C})$ = finitely many pts

i.e. $(g \circ \tilde{u})^{-1}(Z)$ is finite

- we want to use tight of f

$K \subset \overline{K}$, $\{b_1 \dots b_m\} \cup (g \circ \tilde{u})^{-1}(z)$

se poi ho converg. su $\overline{K}$ ce l'ho anche su K

$r' \geq \max\{r, *\}$

$u = g \circ \tilde{u} : \Omega \to X$

(a) Determine $r' \geq 0$

(b) $\exists\ s_n : \Omega \longrightarrow \mathcal{X}$ s.t.

$$\text{TIGHT} \begin{cases} j^{r'} s_n(x_i) = j^{r'} u(x_i) \\ x_i \in \{b_1 \dots b_m\} \cup (g \circ \tilde{u})^{-1}(z) \\ + \quad s_n|_{\overline{K}} \longrightarrow u|_{\overline{K}} \end{cases}$$

(c) Valuative criterion of properness

and obtain

$\tilde{s}_n$ s.t.

$u \circ \tilde{s}_n = s_n$ and

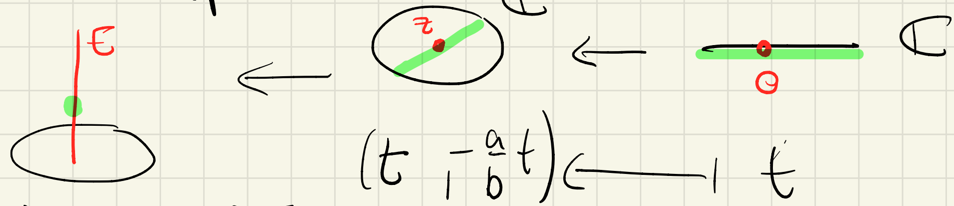
$\tilde{s}_n|_{K} \longrightarrow \tilde{u}|_{K}$

with same jets

Supposons cas simple $Bl_pV \longrightarrow V$

example

a



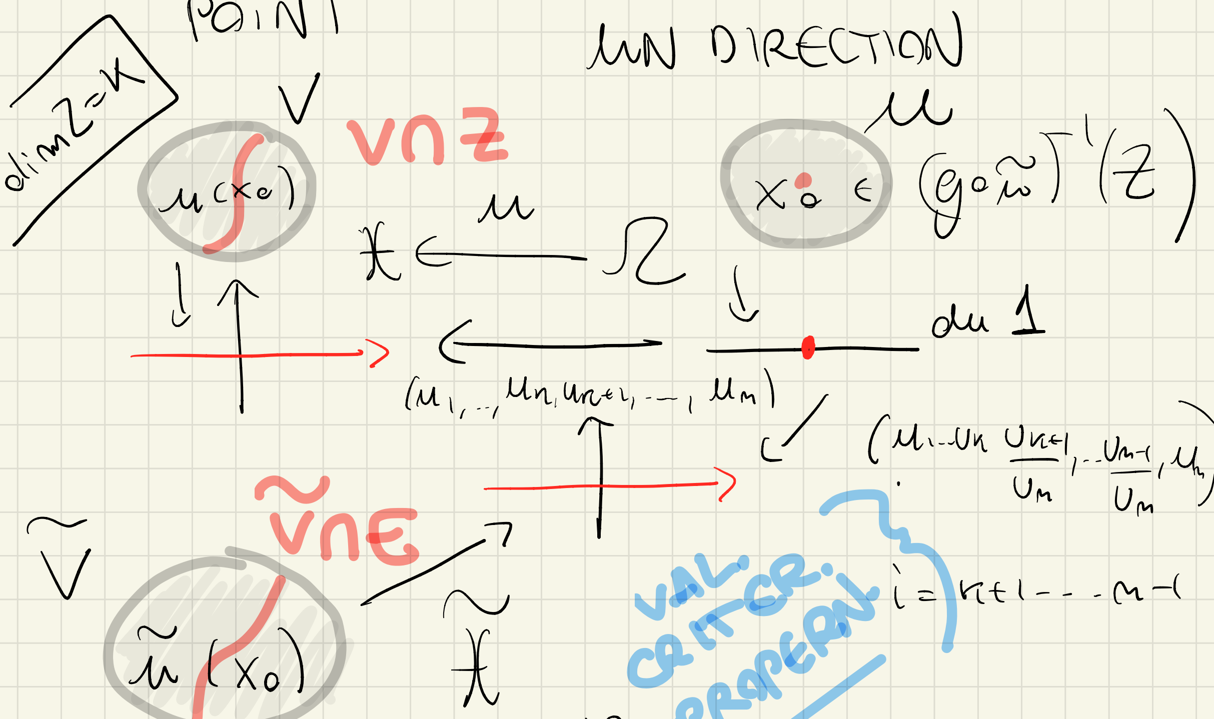
$(t, -\frac{a}{b}t) \longleftarrow\!\!\shortmid\ t$

$(t, -\frac{a}{b}t)[1 : -\frac{a}{b}] \longleftarrow$

$l : ax+by=0$

$(t, -\frac{a}{b}t) = t(1, -\frac{a}{b})$

PASSAGE PAR UN POINT $\longleftarrow\!\!\shortmid$ PASSAGE PAR UN POINT AVEC UN DIRECTION



COORD. LOCALI DEL BLOW-UP

$r' \geq \max\{p+1, r\}$

$M_{\tau} =$

$a_p z^p + \cdots$

$$\mathbb{R} \times S^1 \longrightarrow \mathbb{R}^2$$

$$(t, \theta) \longmapsto (t\cos\theta, t\sin\theta)$$

$t \neq 0$ ISO

$t = 0$

$(0, 0)$

S_1 direzioni

Real alg geo

$\mathbb{P}(\mathcal{N}_{\mathbb{C}^n / \mathbb{C}^n})$

$$\mathbb{C}^k = \left\{ \begin{matrix} x_1 = 0, \ x_2 = 0 \dots \\ x_k = 0 \end{matrix} \right\} \subset \mathbb{C}^n$$

codim $n-k$

coordinate

$y_{k+1} \dots y_n \in \mathbb{P}^{n-k-1}$

localmente

$$\boxed{\tilde{\mathbb{C}}^n = \left\{ \underset{i,j \in \{k+1, \dots, n\}}{x_i y_j = x_j y_i} \right\} = Bl_{\mathbb{C}^k} \mathbb{C}^n}$$

Consideriamo la carta di $\mathbb{P}^{n-k-1}$: $U_n = \{ y_n \neq 0 \}$

$$\mathbb{C}^{n-k} \times \mathbb{C}^k$$

$$(u_1 \dots u_n) \longleftarrow \mathbb{C} \ \underset{x_0}{\cdot}$$

Descrivi localmente il blow-up di V in $\mathbb{C}^k$ (che corrisponde a $V \cap Z$), se esprimi $(u_1 \dots u_n)$ nel blow-up

$$\left\{ y_j = \frac{u_j}{u_n} \cdot y_n \quad \forall j \right\} \cap U_n$$

$$= \left\{ y_j = \frac{u_j}{u_n} \quad \forall j \right\}$$

quindi ottieni $\left(u_1 \dots u_k, \frac{u_{k+1}}{u_n}, \dots, \frac{u_{n-1}}{u_n}, 1 \right)$

$M_{r+1}, \dots, M_m$

f_m has the smallest power in the Taylor expansion z^p

$\frac{M_{r+1}}{M_m}, \dots, \frac{M_{m-1}}{M_m}, M_m$

xche sufficiente ?

$r' \geq \max\{p+1, r\}$

(b) hypothesis f tight

(c) ~~Valutation crit. prop~~ ✓

Explanation about $\mu(\Omega) \cap E$:

<u>Proposition</u> :

$$\begin{tikzcd} Z \arrow[r, "f"] & Y \arrow[l, bend left, "\mu"] \end{tikzcd}$$

f G-equiv. holom. map of cx manifolds

μ G-equiv. holom. section of f

<u>Y STEIN</u>

Suppose Y no isolated pt.

$K \subset Y$ compact G-stable

$\boxed{S} \subset Z$ nowhere dense (i.e. il n'est pas dense dans aucun ouvert de Z)

play role of the exceptional divisor

$b_1 \,---\, b_m \in K$

$r \geq 0$

Then $\exists$ G-stable open $Y' \supset K$

$\exists\, u_n : Y' \longrightarrow Z$ G-equiv. holo

of f above Y' + jets as u

at the b_i and

$$u_n|_K \xrightarrow{\text{unif.}} u|_K$$ and

NO CONNECT. COMPONENTS of $u_n(Y')$ is included in S.

<u>Preuve</u>:

vector bundle $E \supseteq Z$ as neigh. of $u(Y)$

$$\begin{tikzcd} E \arrow[d, "f"', shift right] \\ Y \arrow[u, bend right, "u"'] \end{tikzcd}$$

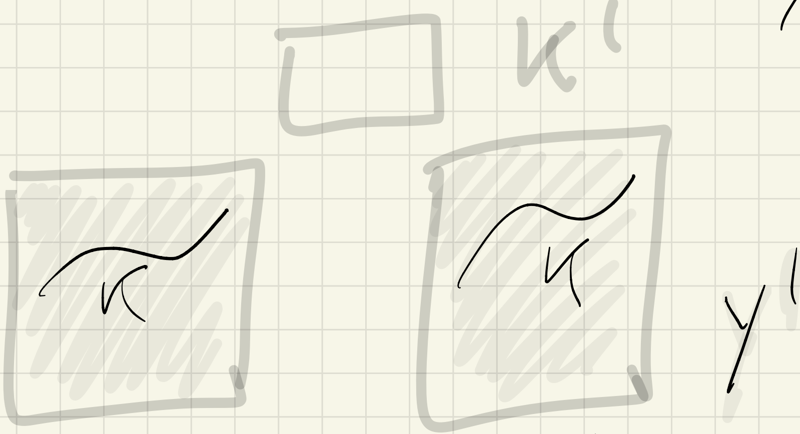
u = zero section

Olivier^2 : $U' \supseteq u(Y)$ as zero sect. $N_{u(Y)/Z}$

G-equiv. biholo $\cong$

$U \supseteq u(Y) \subset Z$

donner un element de $H^0(Y,G)/p$



K compact $\Longrightarrow Y'_1 \dots Y'_e$

$\Sigma = \{ x \in Y : S$ contient un ouvert contenant $u(x)$ in $f^{-1}(x) \}$

Y^G $\Sigma \cup Y^G$ aucun part dense en Y

$y_j \in Y'_j \quad \forall j = 1, \dots, e \quad$ t.q

- $y_j \notin Y^G \cup \Sigma$
- $y_j, \sigma(y_j) \neq b_i$

Comme $y_j \notin \Sigma$ $\exists$ $z_j \in f^{-1}(y_j)$
$\subset E_{y_j}$ t.q. $t z_j \notin S$ octkc1

Y STEIN $\Rightarrow$ $\exists$ $\zeta \in H^0(Y, E)$
qui disparait à l'ordre r sur b_i
t.q. $\zeta(y_j) = z_j$
$$\zeta(\sigma(y_j)) = \sigma(z_j)$$

$$\zeta \rightsquigarrow \frac{\zeta + \sigma(\zeta)}{2} \in H^0(Y, E)^G$$

K' compact, $\zeta/m \rightsquigarrow u_m$

$u_m \longrightarrow$ zero $= u$ sur Y'
pour $m \gg 0$.

✓

Y STEIN ; E is spanned / determined by its global sections

$H^i(Y, E) = 0 \quad \forall i > 0$

E as $\mathcal{O}_Y$-module

$H^0(Y, E)$ as $H^0(Y, \mathcal{O}_Y)$-module

$$H^0(Y, E) \xrightarrow{\text{SURJ.}} \prod_i H^0(Y, E) / P_i$$

P_i ideale max associato a un pt $p_i \in Y$

" teorema cinesi dei resti "

STEIN : $H^0(Y, E) / P_i (E) \cong E_p$

* $\forall i$ cela correspond a regarder la fibre E_p de E au dessous du point qui corresponds à p l'ideal maximale

Donner un pt de E_p $\longleftrightarrow$